$$\frac{k}{h} = \frac{T_w - T_0}{(\partial T/\partial x)_w}. \tag{19}$$

Substituting an exponential form for T (5) into (19), the result is

$$L = k/h \tag{20}$$

where L is the exponential decrement and defined as the film thickness.

This discussion is complicated by the fact that h is a function of the distance along the hot surface in the direction of gas flow. A simple theoretical result that can be obtained is[1]

$$h_y = 0.332 N_p^{1/3} k \sqrt{\frac{v_\infty}{uy}} \tag{21}$$

where y is distance measured from the leading edge along the hot surface, the quantity N_p is the dimensionless Prandtl number of the gas, u is the kinematic viscosity term and v_∞ is the free stream velocity. From (20) and (21), we obtain an equation for L,

$$L = \frac{3}{N_p^{1/3}} \sqrt{\frac{uy}{v_\infty}}. \tag{22}$$

Acknowledgment

The authors wish to acknowledge the advice and assistance of Prof. G. Bekefi in this work. In addition, we acknowledge the assistance of C. Buntschuh in carrying out portions of the analysis and Miss M. Pennell for the computer programming.

The Application of the Focussed Fabry-Perot Resonator to Plasma Diagnostics*

R. I. PRIMICH†, MEMBER, IEEE, AND R. A. HAYAMI†, MEMBER, IEEE

Summary—**The use of a focussed Fabry-Perot resonator at microwave frequencies for plasma diagnostics is discussed. It is shown theoretically that improvements in sensitivity in the measurements of the properties of transparent plasmas and dielectrics of two to three orders in magnitude can be expected. Losses have been neglected. It is indicated that under certain circumstances, refractive index changes in the gaseous environment may be significant. A method of measuring these changes is included. The extension of these techniques to the optical part of the spectrum is discussed and is shown to be promising. Experimental results, obtained with the use of a cavity at 70 Gc, are presented and appear to confirm the main predicted features.**

Introduction

CONVENTIONAL, high-frequency, focussed probes have been successfully used to measure the ionization in wakes behind projectiles in flight at hypersonic velocities, where the wake diameters are comparable to the wavelength.[1] For small diameter wakes, short wavelengths are essential and the resulting lower limit to the minimum measurable electron density at these wavelengths may be too high for some applications. The use of lower frequencies would produce measurable effects at lower densities, but the degradation of resolution might offset this advantage. Since a rigorous solution of the scattering problem (one which would allow for the finite sizes of the beam and wake) is not always practical, there is a need for a technique which retains the resolution of the shorter wavelength probes but provides several orders of magnitude improvement in sensitivity. A system that may provide these features is described.

The basic limitation in sensitivity of a focussed microwave probe lies in the fact that the beam is transmitted only once through the plasma of interest. For a fixed frequency, the limit is reached when the plasma properties are such that the phase shift of the transmitted wave is reduced to the minimum measurable level. This is illustrated in the idealized case of normal incidence of a uniform plane wave on a uniform plasma slab.

For simplicity, the electron collision frequency is assumed to be zero and the plasma to be underdense; that is, $(\omega_p/\omega)^2 \ll 1$ where

ω is the operating frequency
ω_p is the plasma frequency of the ionized medium.

The phase shift of the transmission coefficient caused by the introduction of the slab in the beam can be shown to be[2]

* Received May 21, 1963; revised manuscript received August 27, 1963. This work was supported by General Motors Corporation.

† General Motors Defense Research Laboratories, Santa Barbara, Calif.

[1] R. I. Primich and R. A. Hayami, "Millimeter Wavelength Focused Probes and Focused Probes and Focused, Resonant Probes for Use in Studying Ionized Wakes behind Hypersonic Velocity Projectiles," presented at the Millimeter and Submillimeter Conf., Orlando, Fla.; January 8–10; 1963.

[2] W. M. Cady, M. B. Karelitz and L. A. Turner, "Radar scanners and radomes," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., p. 354; 1948.

$$\phi = -\frac{1}{2} k_v d_e \frac{n_e}{(n_e)_c} \quad (1)$$

from which

$$(n_e)_{\min} = \frac{(n_e)_c \lambda_v}{\pi d_e} \mid \phi \mid_{\min}. \quad (2)$$

ϕ is the measured phase difference, after and before insertion of the slab.
$k_v = 2\pi/\lambda_v$, λ_v is the free space wavelength.
d_e is the thickness of the slab.
n_e is the electron density within the slab.
$(n_e)_c$ is the electron density which would result in cutoff at the frequency.
$(\ )_{\min}$ indicates the minimum measurable quantity within parenthesis.

Practical experience has shown that $|\phi|_{\min}$ for a 70-Gc focused probe is about 0.005 radian. For

$$\frac{d_e}{\lambda_v} = 1, \qquad (n_e)_{\min} \doteqdot 10^{11} e/cm^3.$$

For

$$\frac{d_e}{\lambda_v} = 10, \qquad (n_e)_{\min} \doteqdot 10^{10} e/\text{cm}^3.$$

For some applications, it would be desirable to measure densities as low as 10^7–10^8 e/cm^3 while maintaining the resolution of the 70-Gc probe (about $\frac{1}{4}$ inch).[1]

If, instead of passing once through the plasma, the wave were made to pass through several times, the total phase shift would be cumulative and the improvement in sensitivity proportional to the number of passages. A method by which this might be effected has been proposed[1] and is shown in schematic form in Fig. 1.

If a high Q spherical cavity is centered about the focal point of the focussed lens system, then two features might be expected. First, multiple reflections within the cavity will have the net effect of causing the wave to pass through the plasma slab (located at the focal point) many times, thus increasing the minimum measurable phase shift. Second, since the cavity is fed with a focussed wave it is expected that a resonant mode[3] in which the focussing at the focal point is preserved will be excited. These essential features have been confirmed in an experimental system.[1] Two spherical, perforated, metallic shells with a diameter of 10 inches were used as the cavity, and Q's of up to 100,000 were measured at 70 Gc. The transmission loss was found to be about 9–10 db. The results of detailed analysis of the transmission through dielectric slabs (in which the dielectric constant is very close to unity) is included for the purpose of indicating the present limits of nonresonant probe sensitivity for both real dielectrics and plasmas

[3] W. Culshaw, "Resonators for millimeter and sub-millimeter wavelengths," IRE Trans. on Microwave Theory and Techniques, vol. 9, pp. 135–144; March, 1961.

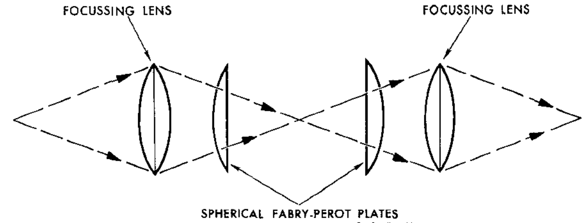

Fig. 1—Layout of focussed resonant cavity.

at very short wavelengths. The analysis is carried out also for an idealized model of the cavity both with and without a dielectric slab. Two possible modes of operation are discussed, and an improvement factor which describes the improvement in sensitivity to be gained by using a cavity is obtained.

Although both dielectric and plasma losses have been neglected, the analysis can be extended to include them. The response of the cavity is, in fact, extremely sensitive to small amounts of loss, which forms the basis of an accurate method of measurement. This will be treated in more detail in a future publication, together with a treatment of cases where the dielectric constant is not close to unity.

It is found that, because of the extreme sensitivity of the cavity to refractive index changes, circumstances will be found in practice in which measurements on ionized gases will include effects due to density changes in the neutral gas. Consequently, a method by which both neutral and electron density changes may be measured is outlined. In addition, the possibility of extending the free space cavity technique to the optical region is discussed. In conjunction with CW lasers, the technique appears to offer significant advances in both neutral gas and electron density measurement.

Finally descriptions of the experimental program and of measurements of the cavity properties are included, together with results of the effect of the ambient air pressure on the cavity.

Determination of the Properties of a Plane-Parallel Dielectric Slab from the Phase Shift of a Plane Wave Transmitted Through It

This analysis is based on the idealized case of a uniform plane wave normally incident on a plane, parallel-sided slab of uniform plasma. For simplicity, the collision frequency is assumed to be zero and the plasma is assumed to be underdense, that is,

$$\left(\frac{\omega_p}{\omega}\right)^2 \ll 1.$$

The plasma slab consists of partially ionized air. The effective refractive index of this medium can be shown to be

$$N_e \doteqdot 1 + \Delta N_e \quad (3)$$

where

$$\Delta N_e \doteq \delta \frac{\rho_2}{\rho_0} - \frac{1}{2} \frac{n_e}{(n_e)_c} \tag{4}$$

δ = constant $\doteq 3\times10^{-4}$ for air (see Kerr[4])
ρ_2 = neutral density of the air within the slab
ρ_0 = atmospheric density.

In this approximation where $\Delta N_e \ll 1$, the effects of neutral gas density and of the free electrons are additive. This would not be true if ΔN_e were large. The result then would be a cross coupling of the two effects.

If the slab is immersed in air at density ρ_1, the phase change in the transmission coefficient caused by the introduction of the slab can be shown to be[2]

$$\phi = k_v N_{01} d_e (\Delta N_e - \Delta N_{01}) \tag{5}$$

where

$$N_{01} = 1 + \Delta N_{01} = 1 + \delta \frac{\rho_1}{\rho_0}$$

$$= \text{refractive index of air at density } \rho_1.$$

This result can be obtained by using the transmission coefficient (t) for a plane-parallel dielectric slab as derived by Cady, *et al*,[2] with appropriate substitutions. If it is assumed that $|\Delta N_e| \ll 1$, then it follows that $|t| \doteq 1$ and arg $t \doteq \phi$ as in (5).

The substitution of (4) into (5) yields

$$\frac{\phi}{2\pi} = \frac{d_e}{\lambda_v} \Delta N = \frac{d_e}{\lambda_v} (\Delta N_1 + \Delta N_2) \tag{6}$$

where

$$N_{01} \doteq 1$$

$$\Delta N_1 = \delta \frac{\rho_2 - \rho_1}{\rho_0}$$

$$\Delta N_2 = -\frac{1}{2} \frac{n_e}{(n_e)_c}.$$

ϕ is essentially the phase change that would be measured in a substitution method. That is, if ϕ_0 were the phase of the wave without the slab, and ϕ_1 were the phase with the slab present, then $\phi = \phi_0 - \phi_1$ where all phases are referred to the second interface.

Eq. (6) is particularly simple. It has been plotted in Fig. 2 which may be used for any of the following, $\Delta N = \Delta N_1 + \Delta N_2$, $\Delta N = \Delta N_1 (\Delta N_2 = 0)$ or $\Delta N = \Delta N_2 (\Delta N_1 = 0)$, to obtain the corresponding ϕ's. An obvious way in which it may be used is to obtain estimates of the mini-

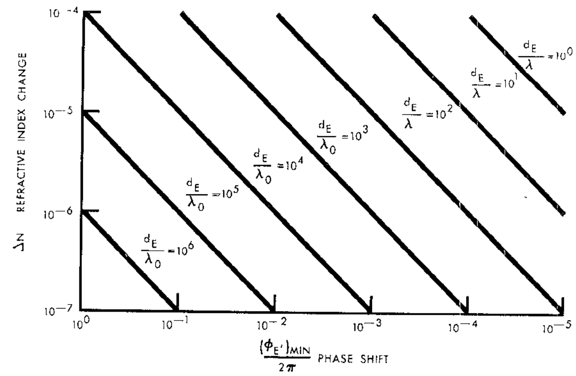

Fig. 2—Phase shift caused by refractive index change in plane slab.

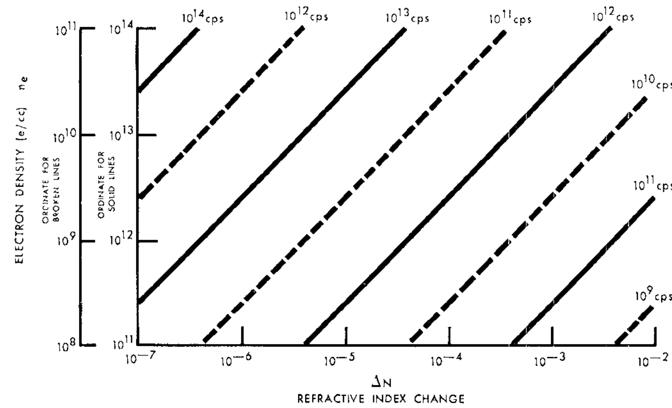

Fig. 3—Electron density and refractive index changes in a slab which produce equal phase shift.

mum n_e that may be measured for specified values of ϕ. For the example given in the Introduction,

$$f = 70G_c, \quad \frac{d_e}{\lambda_v} = 10. \quad \left(\frac{\phi}{2\pi}\right)_{\min} \doteq 10^{-3}.$$

Fig. 2 can be used to obtain

$$(\Delta N)_{\min} = (\Delta N_2)_{\min} = -\frac{1}{2}\frac{n_e}{(n_e)_c} = -10^{-4}.$$

from which $(n_e)_{\min} = 1.2\times10^{10}$ e/cc as before.

The use of Fig. 2 is made even easier if

$$\Delta N_2 = \Delta N = -\frac{1}{2}\frac{n_e}{(n_e)_c}$$

is plotted as in Fig. 3. Here, since $(n_e)_c$ is proportional to f, n_e is plotted against ΔN for various f's. The combined use of Figs. 2 and 3 will result in $(n_e)_{\min}$ directly.

More generally, Figs. 2 and 3 can be used to study the limitations imposed on sensitivity as the operating frequency is changed. Alternatively they can be used to determine the optimum frequency which should be used for a given plasma and receiver sensitivity.

Eq. (6) is so simple that it normally would not be necessary to plot it. However it will be seen later that the sensitivity of the cavity may be referred back to (6) and, for this more general case, Figs. 2 and 3 are useful.

[4] D. Kerr, "Propagation of Short Radio Waves," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., pp. 189–193; 1951.

Determination of Slab Properties Inserted in a Parallel Plate Cavity

The spherical focused resonator which was mentioned in the Introduction has been shown[5] to be theoretically equivalent to a planar Fabry-Perot resonator illuminated by a uniform plane wave. Consequently, an analysis of a plane-parallel plasma slab located between two planar Fabry-Perot reflectors will be useful in the determination of the ultimate performance of a focused resonator which contains a slab of underdense plasma.

The spacing between the plates is d_b and the refractive index of the medium between the plates (in the absence of the plasma slab) is $N_{01}=1+\delta\rho_1/\rho_0$, that is, air at density ρ_1. The transmission coefficient of the cavity, which is partially filled with a thickness d_e of plasma, can be derived using the methods outlined by Cady, *et al.*[2] If, in the derivation, it is assumed that the plasma is underdense ($|\Delta N_e| \ll 1$), the transmission coefficient can be written

$$t = \frac{(1-r^2)}{\sqrt{1-2r^2\cos 2\left\{k_vN_{01}d_0 + k_vN_0d_0\,\frac{d_e}{d_0}\left(\frac{N_e}{N_{01}}-1\right)\right\}+r^4}}\exp -j\left\{k_vN_{01}d_0\,\frac{d_e}{d_0}\left(\frac{N_e}{N_{01}}-1\right)+\psi\right\} \tag{7}$$

and

$$\Psi = \frac{r^2\sin 2\left\{k_vN_{01}d_0+k_vN_{01}d_0\frac{d_e}{d_0}\left(\frac{N_e}{N_{01}}-1\right)\right\}}{1-r^2\cos 2\left\{k_vN_{01}d_0+k_vN_{01}d_0\,\frac{d_e}{d_0}\left(\frac{N_e}{N_{01}}-1\right)\right\}} \tag{7a}$$

where τ = voltage reflection coefficient of either Fabry-Perot plate and $d_0=d_b$ = spacing of plates for resonance. Two possibilities for determining the slab properties from this transmission coefficient follow.

Frequency Shift Method

Prior to insertion of the slab, the cavity is resonated. The resonance condition is

$$2k_vN_{01}d_0 = p\pi, \qquad p = 0, 2, 4, \cdots.$$

If the cavity plates have very high reflectivity with no associated losses, it can be shown that, at resonance, (7) reduces to $|t| \doteq 1$ and $\psi=0$.

The slab is now inserted in the cavity, detuning it, and the resonant frequency (f_0) is adjusted by an amount Δf to restore resonance. The resonance condition is again that the argument of the cosine function in (7) be equal to $p\pi$, that is,

$$2\left\{k_v'N_{01}d_0 + k_v'N_{01}d_0\frac{d_e}{d_0}\left(\frac{N_e}{N_{01}}-1\right)\right\} = p\pi = 2k_vd_0N_0$$

where

$$k_v' = \frac{2\pi}{\lambda_v+\Delta\lambda}.$$

Also,

$$\frac{\Delta\lambda}{\lambda_v} \doteq -\frac{\Delta f}{f_0}.$$

Upon further reduction, it is found that

$$\Delta N = N_e - N_{01} \doteq \delta\frac{\rho_2-\rho_1}{\rho_0} - \frac{1}{2}\frac{n_e}{(n_e)_c} \doteq -\frac{d_0}{d_e}\frac{\Delta f}{f_0}. \tag{8}$$

Now, (1) (for the nonresonant probe) may be written

$$\Delta N = \frac{\phi}{2\pi}\frac{\lambda_v}{d_e} = \frac{\phi}{2\pi}\frac{2\pi N_{01}}{k_vN_{01}d_c}\frac{d_0}{d_e}.$$

Thus, the improvement factor (I.F.) due to the use of the cavity is

$$\text{I.F.} = \frac{\Delta N|_{\text{no cavity}}}{\Delta N_{\text{cavity}}} = \frac{\left(\frac{\phi}{2\pi}\right)\frac{4}{p}}{\frac{\Delta f}{f_0}}. \tag{9}$$

As an example of the practical improvement that might be expected, let

$f_0 = 70$ Gc

$\left(\frac{\phi}{2\pi}\right)_{\min} = 10^{-3}$

$p = 240$ (see experimental work in a later section)

$\Delta f = 7$ kc (easily measured with a stabilized klystron).

Then,

$$\text{I.F.} = 167$$

and

$$(n_e)_{\min} = \frac{1}{167}\times (n_e)_{\min}^{\text{no cavity}} = 7.21\times 10^7 e/cm^3.$$

Eq. (9) has been plotted in Fig. 4 as a function of $(\Delta f/f_0)$ for various $4/p$ and for $\phi/2\pi = 10^{-3}$ (the curves

[5] W. Culshaw, "Further considerations on Fabry-Perot type resonators," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. 10, pp. 331–339; September, 1962.

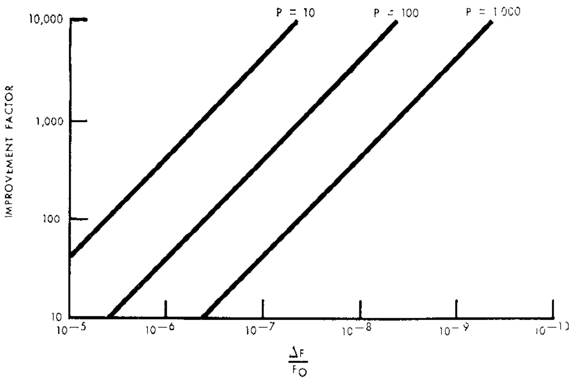

Fig. 4—Improvement factor in sensitivity due to resonant cavity vs frequency shift $\Delta f/f_0$.

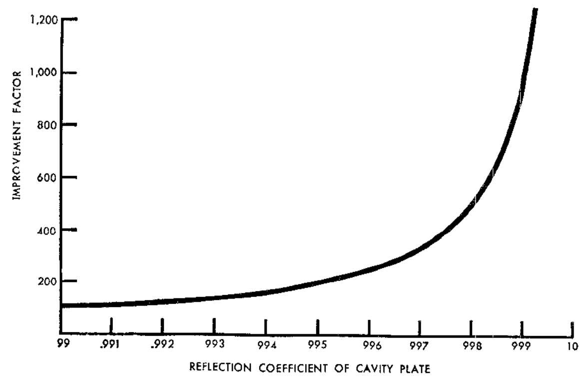

Fig. 5—Improvement factor in sensitivity due to resonant cavity vs reflection coefficient of cavity plates.

are easily scaled for other $\phi/2\pi$). It is apparent (Fig. 4) that large sensitivity improvements are possible.

The Phase Shift Method

If the cavity is only slightly detuned (that is, if the transmission amplitude is still close to unity) by the insertion of the slab, then the change in phase of the transmission coefficient can be shown to be

$$\phi = \left(\frac{p\pi}{2} + Q_0\right)\frac{d_e}{d_0}\left(\frac{N_e}{N_{01}} - 1\right)$$

where the Q of the cavity is given by

$$Q_0 = \frac{p\pi r^2}{1 - r^2} \doteq \frac{p\pi}{1 - r^2} \qquad \text{for } r^2 \doteq 1.$$

It follows that

$$\Delta N \doteq N_e - N_{01} \doteq \delta\frac{\rho_2 - \rho_1}{\rho_0} - \frac{1}{2}\frac{n_e}{(n_e)_c} \doteq \frac{\phi}{2\pi}\frac{2\pi}{\frac{p\pi}{2} + Q_0}\frac{d_0}{d_e}.$$

Therefore,

$$\text{I.F.} = \frac{\Delta N|_{\text{no cavity}}}{\Delta N_{\text{cavity}}} = 1 + \frac{2Q_0}{p\pi} \doteq 1 + \frac{2}{1 - r^2} \doteq \frac{2}{1 - r^2}. \tag{10}$$

In this case the improvement is dependent only upon the quality of the reflector plates.

For the 70-Gc cavity to be described in a later section, $r = 0.9964$ and so I.F. $= 270$. Thus in the example that has been considered in previous sections,

$$(n_e)_{\min} = \frac{1}{270}(n_e)_{\min}^{\text{no cavity}} \doteq 4.5 \times 10^7 e/\text{cm}^3.$$

Eq. (10) has been plotted in Fig. 5 and once again it is apparent that the use of the cavity can result in large improvements in sensitivity.

Effect of Ambient Density on Electron Density Measurements

In many plasmas of current interest, the free electrons are imbedded in a dense neutral gas. If the gas were not ionized, it would have some refractive index other than unity. This could result in a measurable effect in electron density measuring systems, since most of these cannot be used to distinguish electrons from neutral particles. If the gas were partially ionized, two effects might be expected—one due to the neutral density and the other due to the electron density. In some cases, it might be possible to calibrate the system before the onset of ionization to permit the effect of neutral density to be eliminated. However in other situations, such as in the ionized wake behind projectiles moving in air at hypersonic velocities, ionization and neutral density changes occur simultaneously and it is important to determine whether the neutral density changes might be observable.

The refractive index of nonionized air over virtually the entire electromagnetic spectrum is given by[4]

$$N_{01} \doteq 1 + \delta\frac{\rho_1}{\rho_0} = 1 + \Delta N_{01}$$

where ρ_1 is the ambient density, ρ_0 is sea level density and $\delta \doteq 3 \times 10^{-4}$. Over certain parts of the spectrum N_{01} has a loss component. However, it hardly affects the real component given above and will be neglected for present purposes.

The effective refractive index of a slab containing free electrons in air at a certain density ρ_2 surrounded by air density ρ_1 has been discussed in preceding sections. The phase shifts induced by such a slab in either a simple interferometer or a resonant probe have also been determined.

For instance, for the simple probe,

$$\Delta N = \Delta N_1 + \Delta N_2 = \delta\frac{\rho_2 - \rho_1}{\rho_0} - \frac{1}{2}\frac{n_e}{(n_e)_c} = \frac{\phi}{2\pi}\cdot\frac{\lambda_v}{d_e}.$$

It is seen that the phase shift effects of air density and electron density are additive. This is true for ΔN small

($<1/10$), which is easily satisfied for all conditions encountered here.

In preceding sections, the effects of neutral density have been neglected. ΔN, has been assumed to be zero and $\Delta N=\Delta N_2$. It is now pertinent to determine the conditions under which this assumption is justified. For example, let

$$\rho_1 = \rho_0, \qquad \frac{\rho_2}{\rho_0} = \frac{1}{10}.$$

Then

$$\Delta N_1 \doteqdot 2.7 \times 10^{-4}.$$

This situation might be found in the wake behind a hypersonic projectile fired into air at atmospheric pressure. The density in the wake drops, causing a phase shift of the same sign as that due to the electrons. Consequently, if the conditions under which $|\Delta N_1| = |\Delta N_2|$ were determined, then under these conditions the error in n_e would be no more than a factor of two. Since this magnitude of error is considered to be acceptable, a useful criterion for the neglect of neutral gas density would be established, *i.e.*, $|\Delta N_1| \leq |\Delta N_2|$. Fig. 3 can be regarded as a plot of this criterion.

For instance, in the previous example of the nonresonant probe,

$$f = 70\,\text{Gc}, \qquad (n_e)_{\min} = 1.2 \times 10^{10} e/\text{cm}^3$$

and, from Fig. 3,

$$|\Delta N| = |\Delta N_2| = 10^{-4}.$$

For $|\Delta N_1| = 2.7 \times 10^{-4} > 10^{-4}$ as above, neutral gas density changes are just barely perceptible. In this example, $\rho_2 - \rho_1 < 0$. In the case that $\rho_2 - p_1 > 0$ (as in a shock tube), errors could be more serious. For example, when $|\Delta N_1| = |\Delta N_2|$, then $\Delta N = 0$. Thus under conditions for which $(n_e)_{\min} = 10^{10}$e/cm^3, the neutral density might be such that

$$\rho_2 - \rho_1 > 0 \text{ and } \delta\frac{\rho_2 - \rho_1}{\rho_0} = 10^{-4}, \; i.e.,$$

$$|\Delta N| = 10^{-4} \text{ and } \Delta N = 0.$$

The output of the 70-Gc probe would then show zero output—a gross error. Under such conditions a more appropriate criterion would be $|\Delta N_1| \leq 1/10 |\Delta N_2|$.

In the case of the resonant probe, the above considerations still apply except that all ΔN have to be replaced by ΔN/I.F. For example, if I.F. = 100 and $f = 70$ Gc,

$(n_e)_{\min} \doteqdot 10^{10}$ e/cm^3 (nonresonant probe)
$(n_e)_{\min} = 10^8$ e/cm^3 (resonant probe).

Thus

$$\Delta N_2 = -\frac{1}{2}\frac{n_e}{(n_e)_c} = -10^{-6}.$$

So, if no effects due to neutral density are to be encountered for the 70-Gc resonant probe, then

1) ionized wake case

$$|\Delta N_1| \leq 10^{-6}.$$

2) shock tube case

$$|\Delta N_1| \leq 10^{-7}.$$

It is apparent that these ΔN are 1/100 of those with no cavity (as might be expected from the I.F.) and neutral gas density effects are, on the whole, worse in the case of the resonant probe than for the nonresonant probe.

In general Figs. 4 and 5 may be used in conjunction with Figs. 2 and 3 to estimate the effects of neutral gas density.

It has been assumed that neutral gas effects are undesirable. In case it is desired to measure them, this can be done quite simply as follows.

Again

$$\frac{\phi}{2\pi} = \frac{d_e}{\lambda_v}\left(\delta\frac{\rho_2 - \rho_1}{\rho_0} - \frac{1}{2}\frac{n_e}{(n_e)_c}\right).$$

Let ϕ be measured at a wavelength λ_{v1} to yield ϕ_1 and let ϕ_2 be measured independently at a wavelength λ_{v2} to yield ϕ_2.

Then

$$\frac{\phi_1}{2\pi} = \frac{d_e}{\lambda_{v1}}\left(\delta\frac{\rho_2 - \rho_1}{\rho_0} - \frac{1}{2}\frac{n_e}{(n_e)_{c1}}\right)$$

and

$$\frac{\phi_2}{2\pi} = \frac{d_e}{\lambda_{v2}}\left(\delta\frac{\rho_2 - \rho_1}{\rho_0} - \frac{1}{2}\frac{n_e}{(n_e)_{c2}}\right)$$

or

$$\frac{\phi_1 - \phi_2}{2\pi} = -\frac{1}{2}\frac{d_e}{\lambda_{v1}}\frac{n_e}{(n_e)_{c1}} + \frac{1}{2}\frac{d_e}{\lambda_{v2}}\frac{n_e}{(n_e)_{c2}}$$

or

$$n_e = \frac{2\left(\dfrac{\phi_1 - \phi_2}{2\pi}\right)}{\dfrac{1}{\lambda_{v2}(n_e)_{c2}} - \dfrac{1}{\lambda_{v1}(n_e)_{c1}}}$$

which determines n_e. $\delta(\rho_2 - \rho_1)/\rho_0$ can be found by substituting n_e into either equation for ϕ_1 or ϕ_2.

Extension of Microwave Interferometer Techniques to the Optical Spectrum

For the nonresonant probe,

$$\frac{\phi}{2\pi} = \frac{d_e}{\lambda_v}\delta\frac{\rho_2 - \rho_1}{\rho_0} - \frac{1}{2}\frac{d_e}{\lambda_v}\frac{n_e}{(n_e)_c} = K_1\frac{1}{\lambda_v} - K_2\lambda_v$$

where K_1 and K_2 are independent of λ_v. In other words, the neutral density term $\delta(\rho_2-\rho_1)/\rho_0$ is nondispersive and direct benefit is obtained from d_e/λ_v as λ_v decreases. On the other hand, $(n_e)_c\alpha\lambda_v^2$ is highly dispersive and only partial compensation is derived from d_e/λ_v.

In conclusion, neutral density changes are most easily measured at the most suitable shortest available wavelengths and electron distribution changes are best measured at frequencies which are not too much greater than the plasma frequency.

It is interesting to speculate on the possibility of introducing a resonant cavity (say of the confocal type used with CW lasers) into this simple probe. Within the state of the art, reflectivities as high as 0.99 to 0.999 have been achieved; these would result in I.F.'s from about 100 to 1000. Letting

$$\frac{d_e}{\lambda_v} = 10^4 (i.e. f \doteqdot 10^{14} \text{ cps}), \qquad \left(\frac{\phi}{2\pi}\right)_{\min} = 10^{-3},$$

then

$$(\Delta N_1)_{\min} = 10^{-9} \text{ to } 10^{-10}$$

$$(\Delta N_2)_{\min} = 10^{-9} \text{ to } 10^{-10}$$

or

$$(n_e)_{\min} = 1.2 \times 10^{10} \text{ to } 1.2 \times 10^{11} \text{ e/cm}^3.$$

These figures are worth noting since they are orders of magnitude better than anything that has been achieved. In their derivation, two important assumptions have been made, *i.e.*,

1) that $(\phi/2\pi)_{\min}$ as low as 10^{-3} can be measured,
2) that a free space resonator, similar to that to be described, can be built and used in the optical to region.

Regarding 1), it should be noted that $(\phi/2\pi)$ is the usual optical fringe width and, with more conventional incoherent interferometers (such as the Mach-Zender), the minimum fringe shift that can be detected is about 1/10. However, with the advent of the highly stable CW laser, it appears to be within the state-of-the-art to measure down to $(\phi/2\pi)=10^{-3}$.

Regarding 2), the successful operation of the CW laser itself is a demonstration that high quality free space resonators[6] of the confocal type can be built and used. Their use in the optical region should result in significant improvements in measuring low electron densities.

It is proposed that the use of optical techniques indicated above should be seriously pursued. Even if the predicted performance [*i.e.*, $(n_e)_{\min}=10^{10}$ to 10^{11}e/cm^3] is not achieved, a figure of 10^{13}e/cm^3 would represent an important achievement in that the current gap in electron density measurements would be eliminated. Present day millimeter wavelength techniques are useful up to about 10^{14}e/cm^3, and optical interferometers have been used down to about 10^{15} e/cm^3. Confocal resonators which appear to have beamwidths of 1–2 mms would also enhance spatial resolution capabilities. Finally, the availability of time-varying signals due to very low density variations in gases would be extremely useful in studies of turbulence.

Experimental Work

The major part of the experimental program has been devoted to demonstrating the feasibility of building a cavity of acceptable quality and to showing that the predicted sensitivity could be achieved in practice.

Cavity and Mounting Construction

The spherical cavity plates were rigidly mounted in a focussed probe section that was normally used for measuring ionized wakes in a ballistic range (see Fig. 1). Fig. 6 illustrates the plates in position. The probe section consists of a short section of pipe, two feet in diameter with focussing lenses inserted in the sides. Each lens (diameter 10.6 inches) is designed to focus energy from a small horn to the smallest possible dimensions in the focal plane which is located midway between the two lens systems. In the normal nonresonant mode of operation, one lens system serves as the transmitting antenna and the other as the receiving antenna. The ionized wake would then be formed in the focal plane and the wake axis would be normal to the beam direction. The focal distance of each lens was made to be 10.6 inches in order to achieve maximum focusing. More than 90 per cent of the beam energy was found to pass through a $\frac{3}{8}$-inch-diameter aperture in the focal plane for a frequency of 70 Gc.

It was felt that this particular mechanical arrangement would be suitable for testing the cavity since, for good focussing, the lenses had to be accurately made and located within the section. The entire section was designed to be evacuated and was sufficiently rigid so that negligible distortion would be experienced during evacuation.

The cavity plates were machined out of solid aluminum to follow a spherical contour. Each plate has a thickness of 1/16 inch and the radius of curvature is 5 inches. They are solidly mounted to the lens framework. However, three screw adjustments are available on each plate so that they may be moved axially or tilted slightly. The reflectivity of the plates is determined by drilling a prescribed set of holes through the plates. The dimensions and spacing of the holes were arrived at in an empirical manner. Small planar cavities in which the hole size and spacing were varied over a wide range were placed near the focus of the probe and measurements of Q and insertion loss were made. Near the focus, the contours of constant phase are almost planar and it was found that, for planar cavities with an axial length of about one inch, very good results were

[6] A. Yariv and J. P. Gordon, "The laser," Proc. IEEE, vol. 51, pp. 4–29; January, 1963.

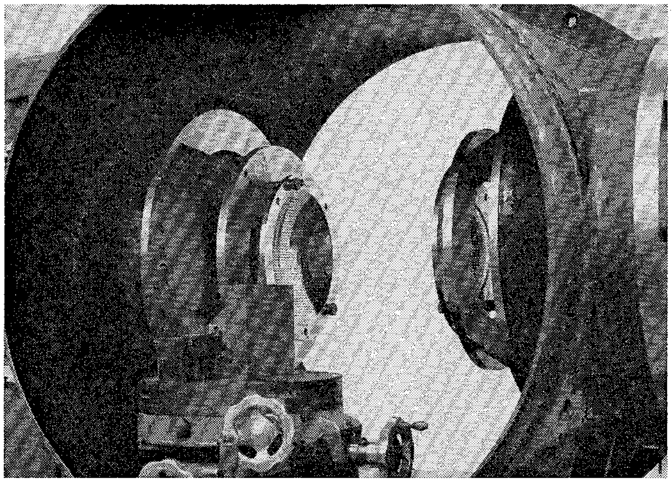

Fig. 6—Photograph of close-up view of resonant cavity in the focussed probe module.

obtained. There were two advantages to using these small cavities: 1) only flat plates were needed and 2) large supplies of flat perforated plates with a variety of hole patterns were commercially available. Once the optimum hole pattern was determined the spherical plates were then prepared according to this pattern.

Measurement of Q and Transmission Loss

The Q of the cavity was measured by sweeping the frequency of the klystron source and displaying the output on an oscilloscope in which the time base was synchronized with the sweep frequency. Then the time base was calibrated by comparing it to a calibrated wavemeter and by beating the output with a frequency stabilized CW source. The width of the Q curve could be measured easily. This technique was used also to check detuning of the cavity.

As the cavity length was varied, a number of different modes occurring at every $\frac{1}{4}$-wavelength change in spacing were noted. They are believed to be the normal axial modes of the nonconfocal geometry.[6] The Q's and transmission losses of these modes varied. However, at a setting of the plates which corresponded closely to the predicted setting, a very strong axial mode was deserved. The Q value was found to be about 80,000–100,000 and the transmission loss was in the range 9–10 db. This is believed to be the desired focussed mode. All other measurements were carried out with this mode in operation. The variation in Q is attributed partly to the difficulty of measurement and partly to coupling from outside the cavity. For instance, the positions of the feed and receiver horns were found to affect the Q. All adjustments were made to maximize the Q.

The smallness of the transmission loss is gratifying—in fact, for the order of frequency and Q indicated above, this is at least as good as the best achieved. The transmission loss is due to the inefficiency with which the incoming wave is coupled to the cavity, resistive losses in the plates and diffraction losses. In this case, the coupling efficiency is believed to be extremely good in that the transverse distribution of the illuminating wave is close to that of the excited mode. Consequently, most of the transmission loss is thought to come from resistive losses in the plates and diffraction loss. This is encouraging in that it may be possible to increase the Q still further. Nevertheless, it was decided to use this cavity, postponing what promised to be extensive work on improving it.

Distribution of Energy in the Focal Plane

The distribution of energy in the focal plane is of great interest since the effective spatial resolution of the resonant probe will depend directly on it. In several unsuccessful attempts made to measure this distribution, it appears that the cavity mode structure was seriously perturbed so that the results that were obtained are mostly meaningless. In spite of this, these measurements will be briefly described and the results interpreted with strong qualifications.

The focal plane field distribution in the nonresonant probe was measured quite simply by passing a small metallic sphere 1/32-inch diameter or about $\lambda/5$ diameter at 70 Gc through the focus in the focal plane. The change in transmission was then interpreted to yield the focal plane field distribution which showed good agreement with that predicted for an appropriately illuminated circular aperture.

When this technique was applied to the resonant probe, both Q and transmission decreased to zero. Further attempts using smaller spheres were equally unsuccessful because, when the sphere was small enough to provide negligible attenuation, the variation in transmission as a function of sphere position was barely discernible. The field distribution in a case in which the perturbation was not too large is shown in Fig. 7. If allowance is made for the size of the probing sphere, the half power width is about $\frac{1}{8}$-inch or about half that for the nonresonant probe. A special feature of these measurements is that in all cases the "skirts" of the distribution were monotonic, showing no evidence of side lobe structure which would be seen in the absence of the resonator and indicating that the cavity was oscillating in a mode of the type discussed by Yariv and and Gordon.[6]

Additional experiments were carried out by placing a metallic screen in the focal plane and observing the transmission as a function of the diameter of a hole in the screen centered at the focal point. Virtually no transmission was observed with an opening of $\frac{1}{4}$-inch diameter, while full transmission was observed for an opening of $\frac{1}{2}$ inch. The change from one condition to the other was almost discontinuous.

Related experiments included the placing of metallic discs of various diameters in the focal plane. In this case, the reflection from the cavity was observed. Here, again, a drastic change took place; resonance was established quite suddenly as the disc diameter was increased from $\frac{1}{4}$ inch to $\frac{1}{2}$ inch.

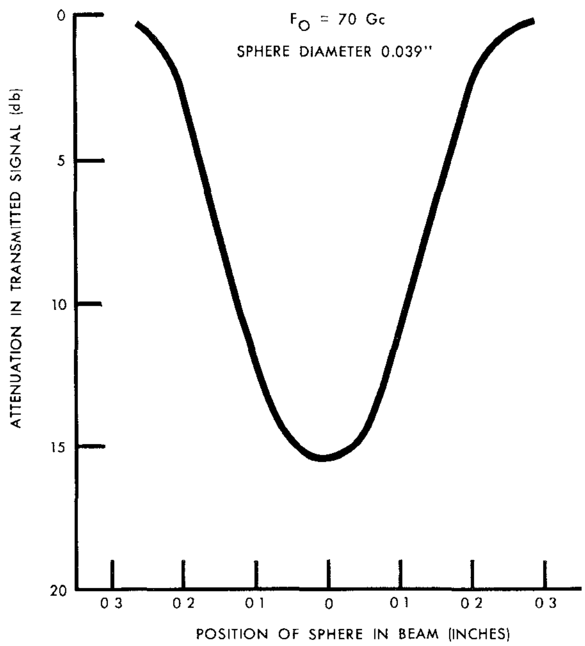

Fig. 7—Field distribution in the focal plane of the focussed resonant cavity.

Finally, it was discovered that a long metallic wire of about 0.010 inch in diameter, placed at any orientation in the focal plane, would reduce the transmission to zero.

Two conclusions may be drawn from these experiments: the major part of the energy in the focal plane lies within a $\frac{1}{2}$-inch-diameter circle and perturbations produced by the measurement systems have been too great to permit a finer definition of the energy distribution. In fact, it may appear that the perturbations are not simply related to the field at the point(s) at which perturbations are produced. If this is the case, it may be more reasonable to refer to Slater's Theorem[7] in which the fields are related to the detuning effects produced by the perturbation rather than to the change in transmission. It is quite apparent that more investigations are required in this area.

Measurement of Cavity Sensitivity

The sensitivity of the cavity to refractive index changes was evaluated in two stages. Initially resonant frequency changes induced by "large" refractive index changes were measured by using real dielectrics, then "small" refractive index changes were studied by varying the air pressure within the cavity.

From (8),

$$|\Delta N| = \frac{d_0}{d_e}\frac{\Delta f}{f_0}.$$

In the case of a real dielectric,

$$N = 1 + \Delta N = \sqrt{\epsilon} = \sqrt{1+\Delta\epsilon} \doteqdot 1 + \tfrac{1}{2}\Delta\epsilon, \quad \text{for } \Delta\epsilon \text{ small.}$$

[7] J. C. Slater, "Microwave Electronics," D. Van Nostrand Co., Inc., New York, N. Y.; 1954. See p. 80.

Let $|\Delta N| = 0.015$ for, say, polyfoam and let

$$d_e = 1 \text{ inch.}$$

Since

$$d_0 = 10 \text{ inches}$$

then

$$\frac{\Delta f}{f_0} = 1.5 \times 10^{-2} \times \frac{1}{10} = 1.5 \times 10^{-3}$$

or

$$\Delta f = 1.5 \times 10^{-3} \times 7 \times 10^4 \text{ Mc.}$$
$$= 105 \text{ Mc}$$

which is a large frequency change in terms of the minimum detectable change (about 7 Kc). The above formula was checked by measuring the frequency change for various lengths of polyfoam. The deduced ΔN was constant within 1 to 2 per cent (for a given sample) and agreed with the nominal value within the same accuracy.

This illustrates in a very practical way the extreme sensitivity of the resonant cavity.

As a further example, a strip of scotch tape placed in the focal plane resulted in a frequency shift of about 2 Mc. In this case, the formula is not strictly valid since $N \doteqdot 1.6$; but, for $d_e = 5 \times 10^{-3}$ inches, the computed shift is 2.1 Mc. This suggests the possibility of using the resonator in the reflection mode to measure the properties of thin films. By coating a film of known thickness on an accurately ground metallic flat, the refractive index could be measured. Alternatively, for a known refractive index, the thickness or variation of thickness could be measured. If $\Delta N = 0.6$, then (with some reservations)

$$(d_e)_{\min} = 1.6 \times 10^{-7} \text{ ins, for } d_0 = 10 \text{ ins} \quad \text{and} \quad \frac{\Delta f}{f_0} = 10^{-7}.$$

These figures reveal that with commonly available materials it is a difficult task to check the sensitivity of the resonant probe for small refractive index change. In order to improve this situation, the cavity was sealed and pumped down to an absolute pressure of about 50 μ. As the pressure was varied continuously, selected readings of frequency shift were noted at various pressures (see Fig. 8). Since both gas temperature and volume remain constant, the pressure is proportional to air density. If ρ_0 is atmospheric density and ρ_1 is the density in the cavity, then (8) can be used to predict the frequency shift.

$$|\Delta N| = \delta\frac{\rho_0 - \rho_1}{\rho_0} = \frac{d_0}{d_e}\left|\frac{\Delta f}{f_0}\right|.$$

Now

$$d_0 = d_e.$$

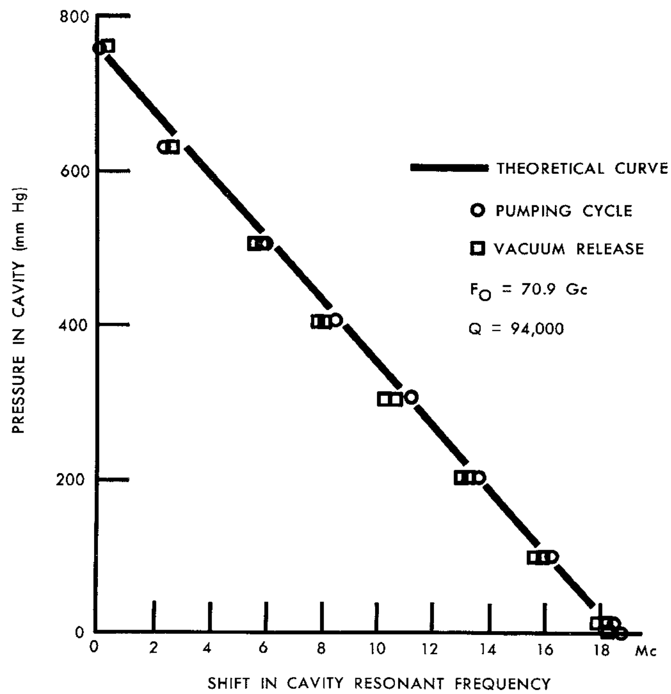

Fig. 8—Plot of pressure vs Δf of the focussed resonator.

So

$$\delta\left(1 - \frac{\rho_1}{\rho_0}\right) = \left|\frac{\Delta f}{f_0}\right|.$$

This theoretical curve is also plotted in Fig. 8. The excellent agreement between theory and experiment is, basically, confirmation of the cavity's sensitivity to small refractive index changes.

Conclusions

It has been demonstrated both theoretically and experimentally that the addition of a resonant cavity of the free space type to a simple interferometer can provide at least two orders of magnitude of improvement in sensitivity. The resonant probe is sensitive to either refractive index changes in gas density or in electron density. In many situations, it may be necessary to use a two frequency system so that gas density and electron density may be resolved separately. The resolution of the resonant probe appears to be at least as good as that without the cavity, but a considerable amount of both experimental and theoretical work is required to define this resolution with more precision.

It is suggested that serious attention be given to the possibility of extending this technique to the optical region. The availability of good CW lasers ensures that significant improvements in measuring low electron densities could be effected.

The cavity technique as outlined is extremely sensitive and accurate in the measurement of the refractive index of small samples. This is of great significance in the direct application to dielectric constant measurement from the millimeter through the optical region. (An extension of the analysis to include losses is required. For small losses this appears to be a straightforward task.) It is also significant to maser applications, in which this type of cavity might serve as the maser cavity.

Additional work is required to utilize this form of cavity in measurements of transient plasma properties. However, in the authors' opinions, this will involve only circuit and operational problems as has already been demonstrated.

Acknowledgment

The authors are particularly indebted to F. Ramsey whose mechanical design and supervision of construction made the difference between a working and nonworking device. In addition, thanks are due various members of the Microwave Laboratory for their critical appraisal of this work.